Received: *May 11, 1999*, Accepted: *June 7, 1999*

Towards a manifestly $\mathrm{SL}(2,\mathbb{Z})$-covariant action for the type-IIB (p,q) super-five-branes

Anders Westerberg and Niclas Wyllard

DAMTP, University of Cambridge,
Silver Street, Cambridge CB3 9EW, UK
E-mail: A.Westerberg@damtp.cam.ac.uk, N.Wyllard@damtp.cam.ac.uk

Abstract: We determine a manifestly SL(2,$\mathbb{Z}$)-covariant κ-symmetric action for the type-IIB (p,q) five-branes as a perturbative expansion in the world-volume field strengths within the framework where the brane tension is generated by a world-volume field. In this formulation the lagrangean is expected to be polynomial; we construct the κ-invariant action to fourth order in the world-volume field strengths.

Keywords: p-branes, D-branes, String Duality, Supersymmetry and Duality.

JHEP06(1999)006

Contents

1. Introduction

Type IIB superstring theory is known to have a non-perturbative SL(2,$\mathbb{Z}$) symmetry [1] under which the p-branes of the theory fall into representations. The strings transform in a doublet ("(p,q) strings") [2, 3], whereas the three-brane is an SL(2,$\mathbb{Z}$) singlet [4]. The five-branes again belong to a doublet as argued in ref. [3]. Supergravity solutions for these (p,q) five-branes were constructed in ref. [5]. In addition, there are seven-branes in the theory which should transform in a triplet under SL(2,$\mathbb{Z}$) [6]. There exist formulations of type IIB supergravity (to which the above branes couple) in which the SL(2) symmetry is manifest [7, 8], a fact that can be exploited to construct world-volume actions for the p-branes of the type IIB theory displaying manifest SL(2) symmetry. This programme has been completed for the strings in refs [9, 10] and for the three-brane in ref. [11]. So far, however, such formulations are lacking for the higher-dimensional branes. It is the purpose of this note to investigate the case of a manifestly SL(2)-covariant action for the five-brane doublet.

The world-volume theory of a (p,q) five-brane should be described on shell by a six-dimensional vector super-multiplet. However, in order to make the SL(2) symmetry manifest, one needs to introduce additional dynamical fields in the action. Our treatment will be within the framework where the brane tension is generated by a world-volume field—in the present case a complex six-form field strength. In this formulation the lagrangean is expected to be a polynomial function of the gauge-invariant world-volume field strengths. To regain the correct counting for the degrees of freedom, auxiliary duality relations are then imposed at the level of the equations of motion. For the three-brane, for instance, a complex world-volume two-form field strength (or two real ones) satisfying a non-linear duality relation is required [11].

In ref. [11] some aspects of a manifestly covariant formulation of a (p,q) five-brane action were presented; we continue this study here using the constructive method of

JHEP06(1999)006

refs [11, 12, 13]. The analysis is complicated by the fact that the tension form is complex and by a "non-canonical" structure of the auxiliary duality relations, forcing us to resort to a perturbative treatment. The main result of our investigations is the determination of the κ-symmetric, manifestly SL(2)-covariant action and the associated projection operator for the type IIB (p,q) five-branes to fourth order in the world-volume field strengths.

In the next section we discuss some facts about the background type IIB supergravity theory and some general features of the world-volume theory of the five-brane doublet. In section 3 we then discuss the method used to construct the action and present our results. Finally, we list our conventions in a short appendix.

2. Preliminaries

The type IIB supergravity theory in ten dimensions [14, 7] is chiral and has a U(1) R-symmetry. In the complex superspace formulation [7] the two Majorana–Weyl spinorial superspace coordinates are combined into a complex Weyl spinor. The theory, furthermore, has an SL(2,ℝ) symmetry at the classical level, which is broken down to SL(2,ℤ) by non-perturbative quantum effects. By gauging the U(1) R-symmetry it is possible to formulate the theory in a way which makes the SL(2) symmetry manifest. In this formulation the scalars of the theory belong to the coset space SL(2,ℝ)/U(1). More precisely, the scalars form a 2×2 matrix

$$\begin{pmatrix} \mathscr{U}^1 & \bar{\mathscr{U}}^1 \\ \mathscr{U}^2 & \bar{\mathscr{U}}^2 \end{pmatrix} \tag{2.1}$$

on which SL(2,ℝ) acts from the left and U(1) acts locally from the right, both group actions leaving invariant the constraint $\frac{i}{2}\epsilon_{rs}\mathscr{U}^r\bar{\mathscr{U}}^s = 1$ (here $\epsilon_{12} = -1$). From the components of the above matrix one can construct the one-forms

$$Q = \frac{1}{2}\,\epsilon_{rs}\,\mathrm{d}\mathscr{U}^r\bar{\mathscr{U}}^s\,, \qquad P = \frac{1}{2}\,\epsilon_{rs}\mathrm{d}\mathscr{U}^r\mathscr{U}^s\,, \tag{2.2}$$

which have special significance [7]. Both are SL(2,ℝ) invariant, whereas under the local U(1) transformation $\mathscr{U}^r \to \mathscr{U}^r e^{i\vartheta}$ they transform as $Q \to Q + \mathrm{d}\vartheta$ and $P \to Pe^{i2\vartheta}$, respectively. Hence, the real one-form Q is a U(1) connection, while P has U(1) charge +2 (the U(1) charge of $\mathscr{U}^r$ is normalised to +1). They furthermore satisfy

$$\mathrm{d}Q - i\,P\wedge\bar{P} = 0\,, \qquad \mathrm{d}P - 2i\,P\wedge Q = 0\,, \tag{2.3}$$

the second equation showing that P is U(1)-covariantly constant (the U(1)-covariant derivative is $\mathrm{D} = \mathrm{d} - ie\,Q$, where e denotes the U(1) charge, and acts from the right). The two physical scalars of the theory are encoded in the projective invariant $\mathscr{U}^1/\mathscr{U}^2$, which in our conventions later will be identified with $-\bar{\tau} = -C_0 + i\,e^{-\phi}$.

JHEP06(1999)006

In addition to the vielbein and the scalars, there are a four-form potential whose field strength is non-linearly self-dual and two two-form potentials. When dealing with p-branes with $p > 3$ one needs to use a formulation of the supergravity theory in which the Poincaré duals to all field strengths are included on the same footing as the original forms. In the present case we therefore also have two six-form potentials (for the seven-branes the eight-form potentials become important too). The SL(2) doublet $\mathscr{U}^r$ discussed above serves as a bridge between quantities transforming in the fundamental representation of SL(2) and SL(2)-invariant (complex) quantities which are charged under the gauged U(1) R-symmetry. Examples include the two- and six-form potentials above, which can be expressed in terms of the SL(2)-invariant forms $\mathscr{C}_2 = \mathscr{U}^r C_{2;r}$ and $\mathscr{C}_6 = \mathscr{U}^r C_{6;r}$, both of U(1) charge +1 (here $C_{2;1} = B_2$, $C_{2;2} = C_2$ and similarly for $C_{6;r}$). We use calligraphic letters to denote complex quantities with U(1) charge +1. Complex conjugation is indicated with a bar and the corresponding quantities have U(1) charge −1. The background field strengths which we need are

$$
\begin{aligned}
\mathscr{H}_3 &= \mathscr{U}^r\, \mathrm{d}C_{2;r}\,, \\
H_5 &= \mathrm{d}C_4 + \frac{1}{2}\,\mathrm{Im}(\mathscr{C}_2\, \bar{\mathscr{H}}_3)\,, \\
\mathscr{H}_7 &= \mathscr{U}^r\, \mathrm{d}C_{6;r} + x\, \mathscr{C}_2\, H_5 - (1-x)\, C_4\, \mathscr{H}_3 + \frac{1}{2}\left(\frac{1}{3} - x\right)\, \mathrm{Im}(\mathscr{C}_2\, \bar{\mathscr{H}}_3)\, \mathscr{C}_2\,,
\end{aligned}
\tag{2.4}
$$

where, following ref. [13], we have introduced a free parameter x in the definition of $\mathscr{H}_7$. These fields satisfy the Bianchi identities

$$
\begin{aligned}
\mathrm{D}\mathscr{H}_3 + i\, \bar{\mathscr{H}}_3\, P &= 0\,, \\
\mathrm{d}H_5 - \frac{i}{2}\, \mathscr{H}_3\, \bar{\mathscr{H}}_3 &= 0\,, \\
\mathrm{D}\mathscr{H}_7 + i\, \bar{\mathscr{H}}_7\, P + \mathscr{H}_3\, H_5 &= 0\,.
\end{aligned}
\tag{2.5}
$$

The constraints which have to be imposed in the superspace approach are at dimension 0

$$
\begin{aligned}
T_{\alpha\bar{\beta}}{}^a = T_{\bar{\alpha}\beta}{}^a &= i\, (\gamma^a)_{\alpha\beta}\,, \\
\mathscr{H}_{a\alpha\beta} &= 2\, (\gamma_a)_{\alpha\beta}\,, \\
H_{abc\bar{\alpha}\beta} = -H_{abc\alpha\bar{\beta}} &= (\gamma_{abc})_{\alpha\beta}\,, \\
\mathscr{H}_{abcde\alpha\beta} &= 2i\, (\gamma_{abcde})_{\alpha\beta}\,.
\end{aligned}
\tag{2.6}
$$

Here the barred indices on the left-hand sides refer to components corresponding to the basis form $E^{\bar{\alpha}} = \overline{E^{\alpha}}$; since barred and un-barred indices are of the same type, the bars have been dropped on the right-hand side (see also the appendix).

For fermionic backgrounds one also needs the dimension-1/2 constraints

$$
\begin{aligned}
P_\alpha &= 2i\, \Lambda_\alpha\,, \\
\mathscr{H}_{ab\bar{\alpha}} &= 2i\, (\gamma_{ab}\, \Lambda)_\alpha\,, \\
\mathscr{H}_{abcdef\bar{\alpha}} &= 2\, (\gamma_{abcdef}\, \Lambda)_\alpha\,,
\end{aligned}
\tag{2.7}
$$

where Λ_α is the dilatino superfield of U(1) charge $+3/2$. The two sets of constraints given above put the theory on shell. Note that we have only displayed the non-vanishing components that are relevant for our calculations. An expedient way to obtain the constraints for $\mathscr{H}_7$ is by translating the results of ref. [15] into the complex formulation used here (taking into account the sign misprint corrected in ref. [13]). The particular choice of gauge we have made use of in converting between the real and complex formulations is $\mathscr{U}^1 = -e^{\frac{1}{2}\phi}C_0 + ie^{-\frac{1}{2}\phi}$ and $\mathscr{U}^2 = e^{\frac{1}{2}\phi}$.

Let us next consider the world-volume gauge field content of the (p,q) five-brane theory. In order to be able to formulate the world-volume action in a manifestly SL(2)-covariant manner one needs the following gauge-invariant world-volume forms:

$$
\begin{aligned}
\mathscr{F}_2 &= \mathscr{U}^r \mathrm{d}A_{1;r} - \mathscr{C}_2\,,\\
F_4 &= \mathrm{d}A_3 - C_4 + \frac{1}{2}\,\mathrm{Im}(C_2\,\bar{\mathscr{F}}_2)\,,\\
\mathscr{F}_6 &= \mathscr{U}^r \mathrm{d}A_{5;r} - \mathscr{C}_6 + x\,\mathscr{C}_2\,F_4 - (1-x)\,C_4\,\mathscr{F}_2 + \frac{1}{2}\left(\frac{2}{3}-x\right)\,\mathrm{Im}(\mathscr{C}_2\,\bar{\mathscr{F}}_2)\,\mathscr{F}_2 + \\
&\quad + \frac{1}{2}\left(\frac{1}{3}-x\right)\,\mathrm{Im}(\mathscr{C}_2\,\bar{\mathscr{F}}_2)\,\mathscr{C}_2\,.
\end{aligned}
\tag{2.8}
$$

The Bianchi identities for these fields are

$$
\begin{aligned}
\mathrm{D}\mathscr{F}_2 + i\,\bar{\mathscr{F}}_2\,P + \mathscr{H}_3 &= 0\,,\\
\mathrm{d}F_4 + H_5 + \frac{1}{2}\,\mathrm{Im}(\bar{\mathscr{F}}_2\,\mathscr{H}_3) &= 0\,,\\
\mathrm{D}\mathscr{F}_6 + i\bar{\mathscr{F}}_6\,P + \mathscr{H}_7 - x\mathscr{H}_3\,F_4 + (1-x)H_5\,\mathscr{F}_2 + \frac{1}{2}\left(\frac{2}{3}-x\right)\,\mathscr{F}_2\,\mathrm{Im}(\mathscr{F}_2\,\bar{\mathscr{H}}_3) &= 0\,.
\end{aligned}
\tag{2.9}
$$

A crucial ingredient of supersymmetric brane actions is κ-symmetry, a local world-volume symmetry for which the variation parameter κ is a target-space spinor satisfying $\kappa = P_+\zeta = \frac{1}{2}(\mathbb{1}+\Gamma)\zeta$, where P_+ is a projection operator of half-maximal rank. It is generally accepted that the background theory being on shell is both a necessary and sufficient condition for κ invariance, although the necessity part has been explicitly proven only in a few cases. We will only investigate κ-symmetry for on-shell backgrounds. The variations of the induced metric and the world-volume form fields under a κ-transformation can be shown to be

$$
\begin{aligned}
\delta_\kappa g_{ij} &= 2\,E_{(i}{}^a\,E_{j)}{}^B\,\kappa^\alpha\,T_{\alpha B}{}^b\,\eta_{ab} + \text{c.c.}\,,\\
\delta_\kappa \mathscr{F}_2 &= -i\,\bar{\mathscr{F}}_2\,i_\kappa P - i_\kappa \mathscr{H}_3\,,\\
\delta_\kappa F_4 &= -i_\kappa H_5 + \frac{1}{2}\,\mathrm{Im}(\bar{\mathscr{F}}_2\,i_\kappa H_3)\,,\\
\delta_\kappa \mathscr{F}_6 &= -i\,\bar{\mathscr{F}}_6\,i_\kappa P - i_\kappa \mathscr{H}_7 + x\,F_4\,i_\kappa \mathscr{H}_3 - (1-x)\,\mathscr{F}_2\,i_\kappa H_5 + \\
&\quad + \frac{1}{2}\left(\frac{2}{3}-x\right)\,\mathscr{F}_2\,\mathrm{Im}(\bar{\mathscr{F}}_2\,i_\kappa \mathscr{H}_3)\,.
\end{aligned}
\tag{2.10}
$$

The next step is to compute the variation of the action under a κ-transformation. On general grounds the action is taken to be of the form

$$S = \int \mathrm{d}^6\xi \, \sqrt{-g}\, \lambda \left[1 + \Phi(\mathscr{F}_2, \bar{\mathscr{F}}_2, F_4) - *\mathscr{F}_6 * \bar{\mathscr{F}}_6\right], \tag{2.11}$$

where λ is a Lagrange multiplier for the constraint $\Upsilon = 1+\Phi(\mathscr{F}_2, \bar{\mathscr{F}}_2, F_4) - *\mathscr{F}_6 * \bar{\mathscr{F}}_6 \approx 0$. For more details on actions of this type, see refs [16, 17, 9, 10, 11, 12, 13]. The function Φ is required to have U(1) charge zero but is otherwise unconstrained at this stage.

It is often convenient to rewrite the action in "form language" as

$$S = \int \lambda \left[*1 + *\Phi(\mathscr{F}_2, \bar{\mathscr{F}}_2, F_4) + \mathscr{F}_6 * \bar{\mathscr{F}}_6\right]; \tag{2.12}$$

this form of the action is better suited for the derivation of the duality relations supplementing it. These relations are constrained by compatibility with the equations of motion encoded in (2.11) and the Bianchi identities (2.9) to take the form [11]

$$\begin{aligned} -2x\, \mathrm{Re}(*\mathscr{F}_6 * \bar{\mathscr{F}}_2) &= K_4 := \frac{\delta\Phi}{\delta F_4}, \\ (1-x) *\mathscr{F}_6 * F_4 + \tfrac{i}{6} *[\mathrm{Re}(*\mathscr{F}_6\, \bar{\mathscr{F}}_2) \wedge \mathscr{F}_2] &= \mathscr{K}_2 := \frac{\delta\Phi}{\delta\bar{\mathscr{F}}_2} \end{aligned} \tag{2.13}$$

(for further details, see refs [11, 12, 13]). These relations are a crucial ingredient in the κ-symmetry analysis to be discussed next. As we shall see, their complicated structure makes this analysis difficult.

3. The method and the result

In this section we determine the action and its associated duality relations from the requirement of κ-symmetry. The analysis is significantly more complicated than for the cases considered previously in the literature as a consequence of the non-canonical structure of the duality relations (2.13) and the fact that the tension form is complex. To make the problem tractable we will use a perturbative approach and expand the action in powers of the field strengths. At first sight it appears that adopting such a procedure would not be possible since there are identities which follow from the duality relations that mix terms of different orders. However, once these identities too are treated in a perturbative order-by-order fashion the procedure becomes consistent.

In order to establish the κ invariance of the action (2.11), it is sufficient to show that the variation of the constraint $\Upsilon = 1 + \Phi(\mathscr{F}_2, \bar{\mathscr{F}}_2, F_4) - *\mathscr{F}_6 * \bar{\mathscr{F}}_6 \approx 0$ vanishes. Using a scaling argument, this variation is found to be

$$\begin{aligned} \delta_\kappa \Upsilon = {} & \left(\bar{\mathscr{K}}_2 \cdot \delta_\kappa \mathscr{F}_2 + \mathscr{K}_2 \cdot \delta_\kappa \bar{\mathscr{F}}_2\right) + K_4 \cdot \delta_\kappa F_4 + \left(\mathscr{F}_6 \cdot \delta_\kappa \bar{\mathscr{F}}_6 + \bar{\mathscr{F}}_6 \cdot \delta_\kappa \mathscr{F}_6\right) - \\ & - \left[\frac{1}{2}(\bar{\mathscr{K}}_l^{(i}\, \mathscr{F}^{j)l} + \mathscr{K}^{(i}{}_l\, \bar{\mathscr{F}}^{j)l}) + \frac{2}{4!} K^{(i}{}_{lmn}\, F^{j)lmn} + \frac{2\cdot 3}{6!} \bar{\mathscr{F}}^{(i}{}_{lmnpq}\, \mathscr{F}^{j)lmnpq}\right] \delta_\kappa g_{ij}\,. \end{aligned} \tag{3.1}$$

By inserting the explicit expressions (2.13) for the K's, as well as the supergravity on-shell constraints given in eqs (2.6) and (2.7), we obtain

$$
\begin{aligned}
(\delta_\kappa \Upsilon)^{(1/2)} = \bar{P}\Bigg[&\Big\{ i \ast\mathscr{F}_6 \ast\gamma_6 - \ast\Big[\ast\mathscr{F}_6\, F_4 + \frac{i}{2}\,\mathrm{Re}(\ast\mathscr{F}_6 \bar{\mathscr{F}}_2)\wedge\mathscr{F}_2\Big]\cdot\gamma_2\Big\}\kappa + \\
&+ \Big\{ i \ast\mathscr{F}_6 \ast [\mathscr{F}_6 - (1-x)\,\mathscr{F}_2\wedge F_4] + \frac{1}{6}\ast\Big[\mathrm{Re}(\ast\mathscr{F}_6\,\bar{\mathscr{F}}_2)\wedge\mathscr{F}_2\wedge\mathscr{F}_2\Big]\Big\}\bar{\kappa}\Bigg] + \text{c.c.} \\
(\delta_\kappa \Upsilon)^{(0)} = 2i\,\bar{E}_i\Bigg[&\Big\{\ast\mathscr{F}_6(\ast\gamma_5)^i - i\ast\Big[\ast\mathscr{F}_6\,F_4 + \frac{i}{2}\,\mathrm{Re}(\ast\mathscr{F}_6\,\bar{\mathscr{F}}_2)\wedge\mathscr{F}_2\Big]^{ij}\gamma_j\Big\}\bar{\kappa} + \\
&+ \Big\{-\frac{i}{6}\,\mathrm{Re}(\ast\mathscr{F}_6 \ast\bar{\mathscr{F}}^{ijkl})\,\gamma_{jkl} + \Big[\Big\{|\ast\mathscr{F}_6|^2 + x\ast\mathrm{Re}(\ast\mathscr{F}_6\,\bar{\mathscr{F}}_2)\wedge F_4\Big\}\,g^{ij} - \\
&- \mathrm{Re}(\ast\mathscr{F}_6\,\bar{\mathscr{F}}_2)^{(i}{}_l \ast F_4{}^{j)l}\Big]\gamma_j - \frac{i}{6}\,\mathrm{Im}\Big[\ast\Big\{\mathrm{Im}\Big(\ast\mathscr{F}_6\,\bar{\mathscr{F}}_2\Big)\wedge\mathscr{F}_2\Big\}^{(i}{}_l\,\bar{\mathscr{F}}^{j)l}\Big]\gamma_j\Big\}\kappa\Bigg] + \text{c.c.}
\end{aligned}
\tag{3.2}
$$

The next step is to insert the projected spinor parameter $\kappa = P_+\zeta$ into these variations using an appropriate Ansatz for the projection operator, and then examine the irreducible components of the expression obtained by expanding the products of γ-matrices (for more details on the method and similar calculations see ref. [13]). It turns out that the parameter x is fixed to the value $2/3$ in the process, a value which corresponds to the field strengths used in ref. [11] after taking into account some differences in conventions. (Actually, it is difficult to conclusively rule out the possibility that x could remain a free parameter; this would, however, require a very intricate form of P_+.)

A major complication of the analysis arises from the fact that, in contrast to all previously considered cases, an overall factor of the tension form can not be factored out from the κ-variation of the constraint. The reason for this can be traced to the fact that there are two linearly independent tension forms ($\ast\mathscr{F}_6$ and $\ast\bar{\mathscr{F}}_6$). This furthermore turns out to lead to the result that one does not get the duality relations in a simple form from any component; rather, one finds the duality relations entangled with various identities implied by them. Although this makes the problem difficult, it is still amenable to a perturbative approach, by means of which we have determined the action and the associated projection operator to fourth order in the world-volume fields. Higher-order corrections to the action, if present, are expected to appear at sixth order only.

The projection operator is found to be

$$
2\ast\gamma_6\,P_\pm\,\zeta = \ast\gamma_6\,\zeta \mp \Big[\frac{2i}{3}\ast F_4\cdot\gamma_2\,\zeta + \frac{i}{3}\ast\mathscr{F}_2\cdot\gamma_4\,\bar{\zeta} + \ast\mathscr{F}_6\,\bar{\zeta}\Big] + \mathcal{O}(F^5)\,, \tag{3.3}
$$

with $\mathcal{O}(F^5)$ denoting terms of total order five in the world-volume field strengths

$\mathscr{F}_2$, $\bar{\mathscr{F}}_2$ and F_4. The final expression for the action is

$$
\begin{aligned}
S = \int \mathrm{d}^6\xi \, \sqrt{-g}\, \lambda \Big[& 1 + \frac{1}{3}\, \mathscr{F}_2 \cdot \bar{\mathscr{F}}_2 + \frac{2}{3}\, F_4 \cdot F_4 + \frac{1}{6}(\beta{-}2) *(\mathscr{F}_2 \wedge F_4) *(\bar{\mathscr{F}}_2 \wedge F_4) + \\
& + \frac{1}{6}(1{-}\beta)\, (\mathscr{F}_2 \wedge \bar{\mathscr{F}}_2) \cdot (*F_4 \wedge *F_4) + \frac{1}{6}\left(\beta{-}\frac{2}{3}\right) \mathscr{F}_2 \cdot \bar{\mathscr{F}}_2\, F_4 \cdot F_4 + \\
& + \frac{1}{6}\,\beta\, (\mathscr{F}_2 \wedge *F_4) \cdot (\bar{\mathscr{F}}_2 \wedge *F_4) + \mathcal{O}(F^6) - *\mathscr{F}_6 * \bar{\mathscr{F}}_6 \Big], \qquad (3.4)
\end{aligned}
$$

which is to be supplemented by the duality relations

$$
\begin{aligned}
-\mathrm{Re}(*\mathscr{F}_6 * \bar{\mathscr{F}}_2) = {} & F_4 + \frac{1}{4}(\beta{-}2)\, \mathrm{Re}\left[*(\mathscr{F}_2 \wedge F_4) * \bar{\mathscr{F}}_2 \right] + \frac{1}{4}(1{-}\beta) *(\mathscr{F}_2 \wedge \bar{\mathscr{F}}_2) \wedge *F_4 + \\
& + \frac{1}{4}\left(\beta{-}\frac{2}{3}\right) (\mathscr{F}_2 \cdot \bar{\mathscr{F}}_2)\, F_4 + \frac{1}{4}\beta\, \mathrm{Re}\left[*(\mathscr{F}_2 \wedge *F_4) \wedge \bar{\mathscr{F}}_2 \right] + \mathcal{O}(F^5)\,, \\
-*\mathscr{F}_6 * F_4 + \frac{i}{2} * \left[\mathrm{Re}(*\mathscr{F}_6\, \bar{\mathscr{F}}_2) \wedge \mathscr{F}_2 \right] = {} & \mathscr{F}_2 + \frac{1}{2}(\beta{-}2)(\mathscr{F}_2 \cdot *F_4) * F_4 + \\
& + \frac{1}{2}\left(\beta{-}\frac{2}{3}\right) (F_4 \cdot F_4) \mathscr{F}_2 - -\frac{1}{2}(1{-}\beta) * [*(*F_4 \wedge *F_4) \wedge \mathscr{F}_2] - \\
& - \frac{1}{2}\beta * [*(*F_4 \wedge \mathscr{F}_2) \wedge *F_4] + \mathcal{O}(F^5)\,. \qquad (3.5)
\end{aligned}
$$

Here β is a free parameter (see below). Two ubiquitous identities in the κ-symmetry calculations are $[\mathscr{F}_2, \bar{\mathscr{F}}_2] = 0$ and $[\mathscr{F}_2, *F_4] = 0$, where $\mathscr{F}_2$, $\bar{\mathscr{F}}_2$ and $*F_4$ are viewed as matrices and the bracket is a matrix commutator. Another important result needed to verify κ-symmetry is the relation

$$
\mathrm{Im}\,(*\mathscr{F}_6 * \bar{\mathscr{F}}_2) = -\frac{1}{6} \mathscr{F}_2 \wedge \bar{\mathscr{F}}_2 + \frac{2}{3} *F_4 \wedge *F_4 + \mathcal{O}(F^6)\,. \qquad (3.6)
$$

These identities can be shown to follow from the duality relations (3.5). They are also required in order for P_+ to have the correct properties. In addition, one needs to use the fact that the following relation holds when the duality relations are satisfied:

$$
\Phi \approx \frac{1}{9} \mathscr{F}_2 \cdot \bar{\mathscr{F}}_2 + \frac{4}{9} F_4 \cdot F_4 + \mathcal{O}(F^6)\,. \qquad (3.7)
$$

This relation follows from the expression for Φ encoded in (3.4) combined with the identity $2x\,\mathrm{Re}\,(\mathscr{K}_2 \cdot \bar{\mathscr{F}}_2) + (1{-}x)\, K_4 \cdot F_4 = 0$ (for $x = 2/3$), which can readily be derived from the form of the duality relations (2.13). Once one has shown that the duality relations imply the above commutator identities, it is straightforward to check that the terms in the duality relations proportional to β vanish for purely algebraic reasons, showing that β can chosen arbitrarily. (It is likely that β will be fixed in the complete action.) Let us also mention that one can change the appearance of the fourth-order terms. For instance, it follows from the duality relations given above that $\mathscr{F}_2 \wedge \bar{\mathscr{F}}_2 - *F_4 \wedge *F_4 = \mathcal{O}(F^4)$; adding this expression squared to the action does not violate κ-symmetry to fourth order.

In order to see that the tension of the (p, q) five-branes described by the action (3.4) works out correctly one proceeds analogously to the discussion in ref. [10]. To get agreement with the formula for the tension first obtained in ref. [3], one has to take into account the fact that when transforming to the string frame the tension receives an additional overall factor $e^{-\phi}$ compared to the string case (recall that $g_{mn}^{\text{string}} = e^{\frac{1}{2}\phi} g_{mn}^{\text{Einstein}}$).

The question arises to what extent the above action differs from the complete action. It appears likely that the modifications, if any, should be rather minor. Furthermore, it is not clear whether P_+ has to be modified (the above expression, obtained from a fairly general Ansatz, certainly looks deceptively simple). However, if (3.3) is the complete result, it seems difficult to modify the action without ruining the property (3.7). It appears that some new input is needed to make further progress; this is especially true in the case of the search for a manifestly SL(2,$\mathbb{Z}$)-covariant formulation of the seven-branes. These are known [6] to form a triplet and couple to the eight-form potentials dual to the three scalars which belong to the SL(2,$\mathbb{R}$)/U(1) coset. Perhaps one way to make further progress is via T-duality; it may be possible to derive T-duality rules which relate duality covariant actions in the M/type IIA and type IIB theories, and in this way make the problem more tractable. It would also be desirable to have a more uniform description of the manifestly SL(2,$\mathbb{Z}$)-covariant type-IIB-brane actions.

Acknowledgments

The work of A.W. and N.W. was supported by the European Commission under the contracts FMBICT972021 and FMBICT983302, respectively. A.W. would like to thank Martin Cederwall for collaboration on the project which initiated the present study.

A. Conventions

We employ a complex superspace notation in which a one-form is expanded in a local inertial-frame basis as $\Omega_1 = E^A\,\Omega_A = E^a\,\Omega_a + E^\alpha\,\Omega_\alpha + E^{\bar\alpha}\,\Omega_{\bar\alpha}$, with $E^{\bar\alpha} = \overline{E^\alpha}$. The relation to the real formulation used in ref. [15] is $E^\alpha = E^{1\alpha} + iE^{2\alpha}$, $E^{\bar\alpha} = E^{1\alpha} - iE^{2\alpha}$, $\Omega_\alpha = \frac{1}{2}(\Omega_{1\alpha} - i\,\Omega_{2\alpha})$ and $\Omega_{\bar\alpha} = \frac{1}{2}(\Omega_{1\alpha} + i\,\Omega_{2\alpha})$. Given these translation rules our spinor conventions follow those of ref. [15]. Moreover, complex conjugation of a bispinor reverses the order of the spinors. For higher forms we use the additional convention $\Omega_n = \frac{1}{n!} E^{A_n} \wedge \cdots \wedge E^{A_1} \Omega_{A_1 \ldots A_n}$. The exterior derivative d acts from the right, so that $\mathrm{d}(\Omega_m \wedge \tilde\Omega_n) = \Omega_m \wedge \mathrm{d}\tilde\Omega_n + (-1)^n \mathrm{d}\Omega_m \wedge \tilde\Omega_n$. (We usually suppress the symbol $\wedge$ when no confusion should arise.) The world-volume forms (which are bosonic) follow the same conventions and hence obey the same rules. Furthermore, we do not distinguish notationally between a target-space form Ω_n and its pull-back to the world-volume, the components of which are given by

$$\Omega_{i_1\ldots i_n} = E_{i_n}{}^{A_n} \ldots E_{i_1}{}^{A_1}\,\Omega_{A_1\ldots A_n} := \partial_{i_n} Z^{M_n}\, E_{M_n}{}^{A_n} \ldots \partial_{i_1} Z^{M_1}\, E_{M_1}{}^{A_1}\,\Omega_{A_1\ldots A_n}\,. \quad (A.1)$$

The Hodge dual of a world-volume n-form is defined by

$$(*\Omega_n)^{i_1\ldots i_{6-n}} = \frac{1}{n!\sqrt{-g}}\epsilon^{i_1\ldots i_6}\Omega_{i_{6-n+1}\ldots i_6}\,, \quad (A.2)$$

where g is the determinant of the induced metric $g_{ij}{=}\partial_i Z^m \partial_j Z^n\, g_{mn}$ (with mostly-plus signature) and $\epsilon^{i_1\ldots i_6}$ is the totally antisymmetric tensor density satisfying $\epsilon^{01\ldots 5} = +1$. World-volume γ-matrices are defined as the pull-backs $\gamma_i = E_i{}^a\,\Gamma_a$. Their symmetrised product obeys the Clifford algebra $\{\gamma_i, \gamma_j\} = 2\,g_{ij}\,\mathbb{1}$ inherited from the target-space, while their antisymmetrised products can be combined into the forms

$$\gamma_n = \frac{1}{n!}\,\mathrm{d}\xi^{i_n}\wedge\cdots\wedge\mathrm{d}\xi^{i_1}\,\gamma_{i_1\ldots i_n}\,, \quad (A.3)$$

where $\gamma_{i_1\ldots i_n} = \gamma_{[i_1}\ldots\gamma_{i_n]}$ and the antisymmetrisation is of weight one. And finally, we use the notation

$$A_n\cdot B_n = \frac{1}{n!}A^{i_1\ldots i_n}B_{i_1\ldots i_n} \quad (A.4)$$

for the scalar product of two world-volume n-forms.

References

[1] C.M. Hull and P.K. Townsend, *Enhanced gauge symmetries in superstring theories. Nucl. Phys.* **B 451** (1995) 525 [hep-th/9505073].

[2] J.H. Schwarz, *An* $\mathrm{SL}(2,\mathbb{Z})$ *multiplet of type IIB superstrings. Phys. Lett.* **B 360** (1995) 13, Erratum *ibid.* **364** (1995) 252 [hep-th/9508143].

[3] E. Witten, *Bound states of strings and p-branes. Nucl. Phys.* **B 460** (1996) 335, [hep-th/9510135].

[4] M.J. Duff and J.X. Lu, *The selfdual type IIB superthreebrane. Phys. Lett.* **B 273** (1991) 409.

[5] J.X. Lu and S. Roy, *An* $\mathrm{SL}(2,\mathbb{Z})$ *multiplet of type IIB super five-branes. Phys. Lett.* **B 428** (1998) 289 [hep-th/9802080].

[6] P. Meessen and T. Ortín, *An* $\mathrm{SL}(2,\mathbb{Z})$ *multiplet of nine-dimensional type II supergravity theories. Nucl. Phys.* **B 541** (1999) 195 [hep-th/9806120].

[7] P. S. Howe and P.C. West, *The complete* $N = 2$, $d = 10$ *supergravity. Nucl. Phys.* **B 238** (1984) 181.

[8] G. Dall'Agata, K. Lechner and M. Tonin, *$D = 10$, $N =$ IIB supergravity: Lorentz invariant actions and duality. J. High Energy Phys.* **07** (1998) 017, [hep-th/9806140].

[9] P.K. Townsend, *Membrane tension and manifest IIB S duality. Phys. Lett.* **B 409** (1997) 131 [hep-th/9705160].

[10] M. Cederwall and P.K. Townsend, *The manifestly* $\mathrm{SL}(2;\mathbb{Z})$ *covariant superstring. J. High Energy Phys.* **09** (1997) 003 [hep-th/9709002].

[11] M. Cederwall and A. Westerberg, *World volume fields,* $\mathrm{SL}(2;\mathbb{Z})$ *and duality: the type IIB three-brane. J. High Energy Phys.* **02** (1998) 004 [hep-th/9710007].

[12] M. Cederwall, B.E.W. Nilsson and P. Sundell, *An action for the superfive-brane in D=11 supergravity. J. High Energy Phys.* **04** (1998) 007 [hep-th/9712059].

[13] A. Westerberg and N. Wyllard, *Supersymmetric brane actions from interpolating dualisations.* hep-th/9904117.

[14] J.H. Schwarz, *Covariant field equations of chiral $N = 2$ $D = 10$ supergravity. Nucl. Phys.* **B 226** (1983) 269.

[15] M. Cederwall, A. von Gussich, B.E.W. Nilsson, P. Sundell and A. Westerberg, *The Dirichlet super p-branes in ten-dimensional type IIA and IIB supergravity. Nucl. Phys.* **B 490** (1997) 179 [hep-th/9611159].

[16] E. Bergshoeff, L.A.J. London and P.K. Townsend, *Space-time scale invariance and the super p-brane. Class. and Quant. Grav.* **9** (1992) 2545 [hep-th/9206026].

[17] E. Bergshoeff and P.K. Townsend, *Super D-branes revisited. Nucl. Phys.* **B 531** (1998) 226 [hep-th/9804011].